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1. FUNDAMENTAL CONCEPTS

1.1 TYPES OF LOADS

- Structural systems may be subjected to one form or another of dynamic loading during their lifetime.
- Two different approaches are available for evaluating the response of a structure to dynamic loads:
  Deterministic
  Nondeterministic
- Deterministic analysis is used when the time varying characteristics of the *prescribed dynamic loads* are well defined. Prescribed loads can be divided in two basic categories:
  Periodic
  Nonperiodic
- Nondeterministic analysis is used when the time variation of the loads is not completely known but can be defined in a statistical sense - *random dynamic loading*.

1.2 BASIC CONCEPTS

- **Newton's Second Law**

  If a force "f" acts on a body which is moving with velocity "v", the rate of change of the quantity of motion of the body (momentum = *mv*) is equal to the applied force.

$$\frac{d(mv)}{dt} = f$$

  In structural dynamics we deal mostly with systems in which the mass "m" is constant. For such systems,

$$f = m\,\frac{d(v)}{dt} = ma$$

- **Degrees of Freedom**
  The number of degrees of freedom in a dynamic system is the least number of coordinates needed to define the position of all the particles of mass in the system.

- **Mass and Weight**
  Mass is a measure of the quantity of matter.

  Weight is a measure of the force necessary to impart a specified acceleration to a specific mass.

  The **acceleration of gravity, g,** is the acceleration that the gravity of the earth would impart to a free-falling body at sea level.

  S.I. units: g = 9.807 m/sec^2.
  English system: g = 32.174 ft/sec^2.

2. SINGLE-DEGREE-OF-FREEDOM (SDOF) SYSTEMS

2.1 UNDAMPED SYSTEM

Consider a single storey building, idealized as an oscillator of mass **m**, and stiffness **k**, which is moving in the direction of the *x*-axis as shown in Fig. 2.1. A single displacement coordinate, **x(t)**, describes the position of all the mass in the system.

The oscillation of the system is initiated by displacing the mass and then releasing it, with no further external force being applied. In this case the mass oscillates back and forth forever since there is no frictional damping to decrease the amplitude of vibration.

The equilibrium equation of the mass is: $\mathbf{F_{inertial} + F_{spring} = 0}$

The spring force is kx(t) and the inertial force is given by Newton's second law, ma(t). Therefore,

$$\mathbf{ma(t) + kx(t) = 0}$$

This is a linear homogeneous second-order differential equation of motion with constant coefficients. The solution of this equation is given by

$$\mathbf{x(t) = A \sin(pt) + B \cos(pt)}$$

- The constants A and B depend on the value of the initial displacement X_0 and of the initial velocity V_0:

  $A = V_0/p$

  and $B = X_0$

  We define: $\mathbf{p = \sqrt{(k/m)}}$ as the ***circular natural frequency*** (rad/sec)

- The circular natural frequency, **p**, linear frequency, **f**, and natural period of oscillation, **T**, are related by

  $$p = 2\pi f = 2\pi/T$$

- The response of the system is a shown in Fig. 2.2 and its maximum displacement is given by

$$x_{max} = \sqrt{(V_0/p)^2 + (X_0)^2}$$

2.2 DAMPED SYSTEMS

All real systems have at least one way of dissipating energy. A damped SDOF system such as the one shown in Fig. 2.3 will vibrate with continuously-decreasing amplitude due to the energy loss during each swing. The frictional force in an ideal SDOF system is proportional to the velocity of the mass, v(t); i.e.,

$$\mathbf{F_{damping} = cv(t)}$$

The differential equation of motion is given by

$$\mathbf{ma(t) + cv(t) + kx(t) = 0}$$

Since $a(t) = \ddot{x}(t)$ is the second derivative of displacement $x(t)$ and $v(t) = \underline{x}(t)$ is the first derivative of displacement, we can rewrite the equation of motion in the following form

$$\mathbf{\ddot{x}(t) + (c/m)\underline{x}(t) + (k/m)x(t) = 0}$$

or

$$\mathbf{\ddot{x}(t) + 2\zeta p\underline{x}(t) + p^2x(t) = 0}$$

in which $2\zeta p = c/m$ and ζ is called the ***fraction of critical damping***. The value of c (and hence ζ) will determine the performance of the system.

If $\zeta = 1$, the system returns to its equilibrium position without oscillating (Critically damped system).

If $\zeta > 1$, the system returns to its equilibrium position without oscillating, but less rapidly than in the case for a critically damped system (Overdamped system).

If $0 < \zeta < 1$, the system returns to its equilibrium position with an oscillatory motion that decays exponentially (Underdamped system).

An example of time histories for each of these cases is shown in Fig. 2.4.

For the case of $\zeta < 1$, the response of the system is given by:

$$x(t) = e_{-}^{\zeta pt} X_0 \cos p_d t + [\frac{V_0 + \zeta p X_0}{p_d}] \sin p_{dt}$$

where $p_d = p\sqrt{(1-\zeta^2)}$ is the *damped circular natural frequency* of the system. The *damped natural period* is thus given as $T_d = 2\pi/p_d$. In this case, the free-vibration motion of the system decays exponentially with time as shown in Fig. 2.5.

2.3 HARMONIC FORCES

The differential equation of motion for a damped system subjected to a sinusoidal excitation of the form $P_0 \sin(\omega t)$ of frequency ω is

$$\mathbf{ma(t) + cv(t) + kx(t) = P_0 \sin(\omega t)}$$

The steady-state response of the system is given by:

$$\mathbf{x(t) = (P_0/k)\, R_{\mathit{d}} \sin(\omega t - \varphi)}$$

where

$$R_d = \frac{1}{\sqrt{[1 - (\omega/p)^2]^2 + [2\zeta\omega/p]^2}} \quad \text{and} \quad \varphi = \tan^{-1} \frac{2\zeta\omega/p}{1 - (\omega/p)^2}$$

R_d is the ***displacement response or dynamic amplification factor*** and φ is the ***phase lag***. The phase lag represents the time it takes the system to react to the excitation once it has reached a steady state motion (see Fig. 2.6). The sensitivity of R_d and φ to variations of the frequency parameter ω/p and damping factor ζ is presented in Fig. 2.7. At resonance (ω/p=1) the largest amplification factors are achieved, but the damping value controls significantly the amplitude of the response.

2.4 TRANSIENT RESPONSE

The differential equation of motion for a damped system subjected to a transient excitation P(t) is

$$\mathbf{ma(t) + cv(t) + kx(t) = P(t)}$$

For a system with zero initial displacement and velocity, its <u>transient</u> response to the exciting force can be expressed in terms of Duhamel's integral:

$$x(t) = \frac{1}{mp_d} \int_0^t P(\tau) e^{-\zeta p(t-\tau)} \sin p_d(t-\tau) d\tau$$

If the initial displacement and velocity are other than zero, the free vibration that they cause should be added to the transient response to obtain the complete response. The transient response of a damped SDOF system subjected to an impulsive force and to a step-like force is shown in Fig. 2.8.

2.5 BASE EXCITATION

In structures subjected to earthquakes the driving force is not an explicit force applied to the mass, but an implicit inertia force. In the system shown in Fig. 2.9, we define the dynamic base displacement as $\mathbf{y_g(t)}$, the displacement coordinate **u(t)** is the displacement of the mass *relative to the base*, and the total displacement of the mass from the position of equilibrium is $\mathbf{x(t) = y_g(t) + u(t)}$.

The restoring force is a function of the *relative displacement u(t)*, the damping force is a function of the *relative velocity, $\dot{u}(t)$*, and the inertia force is a function of the *total acceleration of the mass, ü(t)* Dynamic equilibrium leads to the equation of motion:

$$\mathbf{m\ddot{u}(t) + c\dot{u}(t) + ku(t) = -m\ddot{y}_g(t)}$$

Note that in this case the inertia force $-m\ddot{y}_g(t)$ takes the place of the driving force P(t). An example of the response of a SDOF system with a natural period of 2.5 seconds subjected to a base excitation (earthquake ground motion) is illustrated in Fig. 2.10.

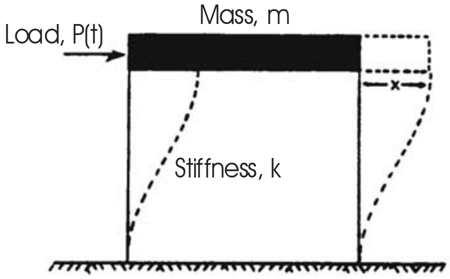


Fig. 2.1 SDOF undamped system.

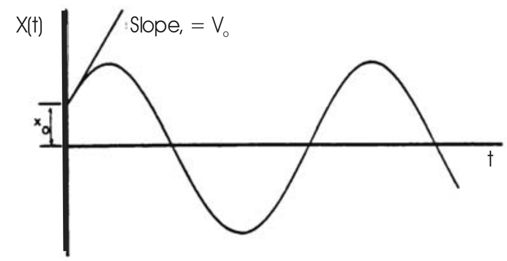


Fig. 2.2 Free vibration of undamped system.

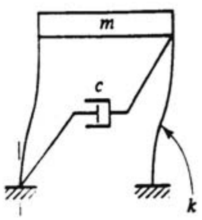


Fig. 2.3 Damped SDOF system.

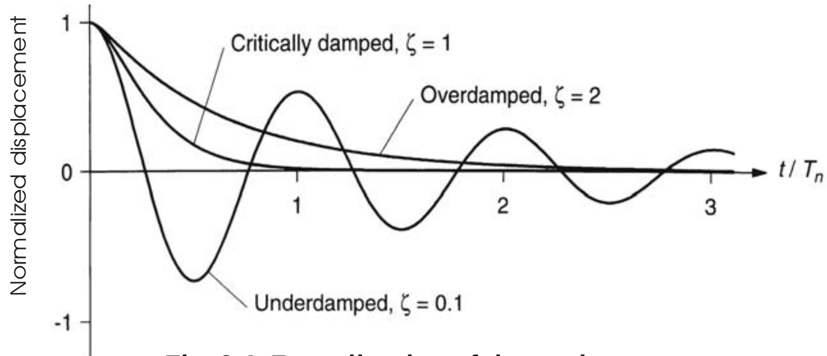


Fig. 2.4 Free vibration of damped systems.

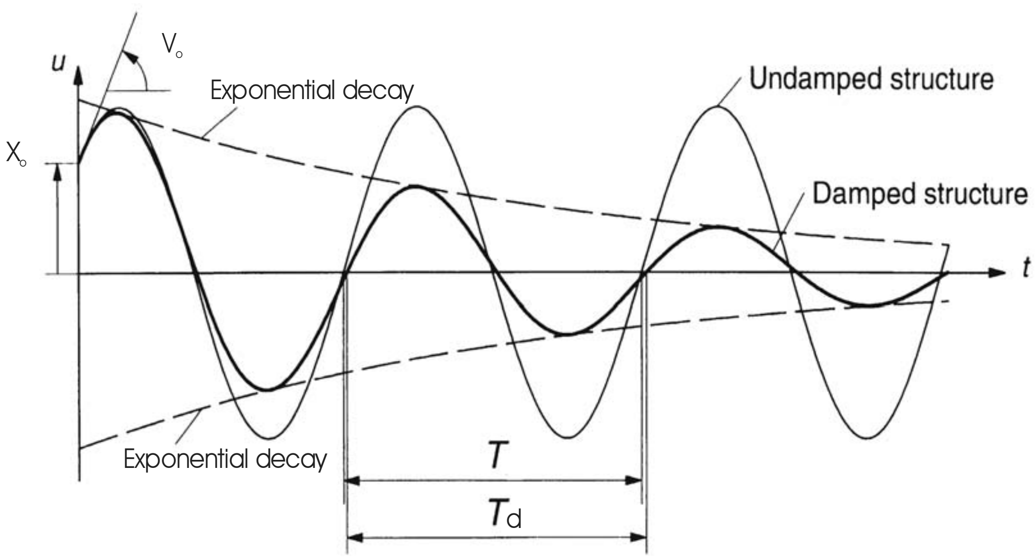


Fig. 2.5 Effects of damping on free vibration of SDOF system.

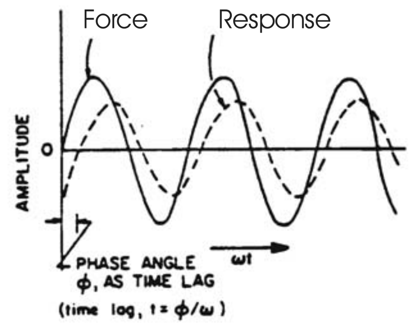


Fig. 2.6 Steady-state response of SDOF system due to harmonic excitation.

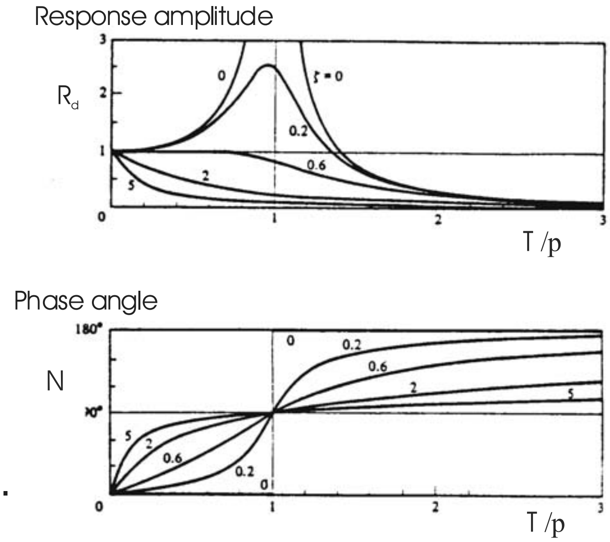


Fig. 2.7 Dynamic amplification factor and phase angle of SDOF systems due to harmonic excitation.

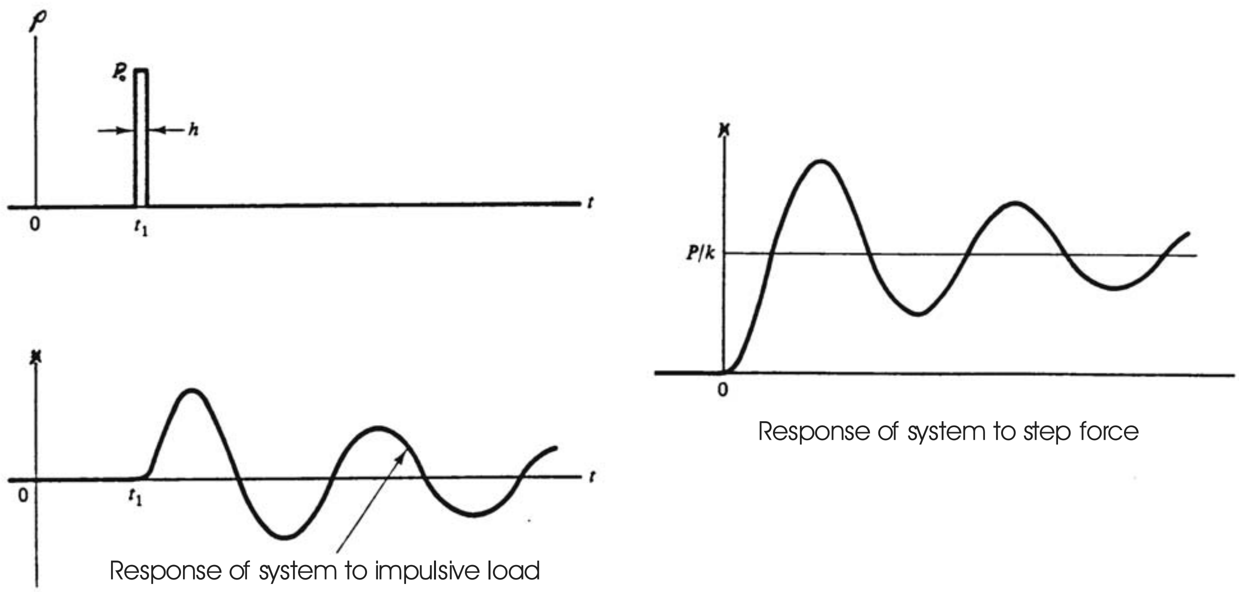


Fig. 2.8 Transient response of SDOF system to simple excitations.

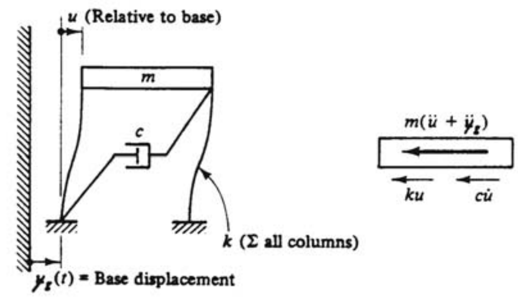


Fig. 2.9 Base excited SDOF system.

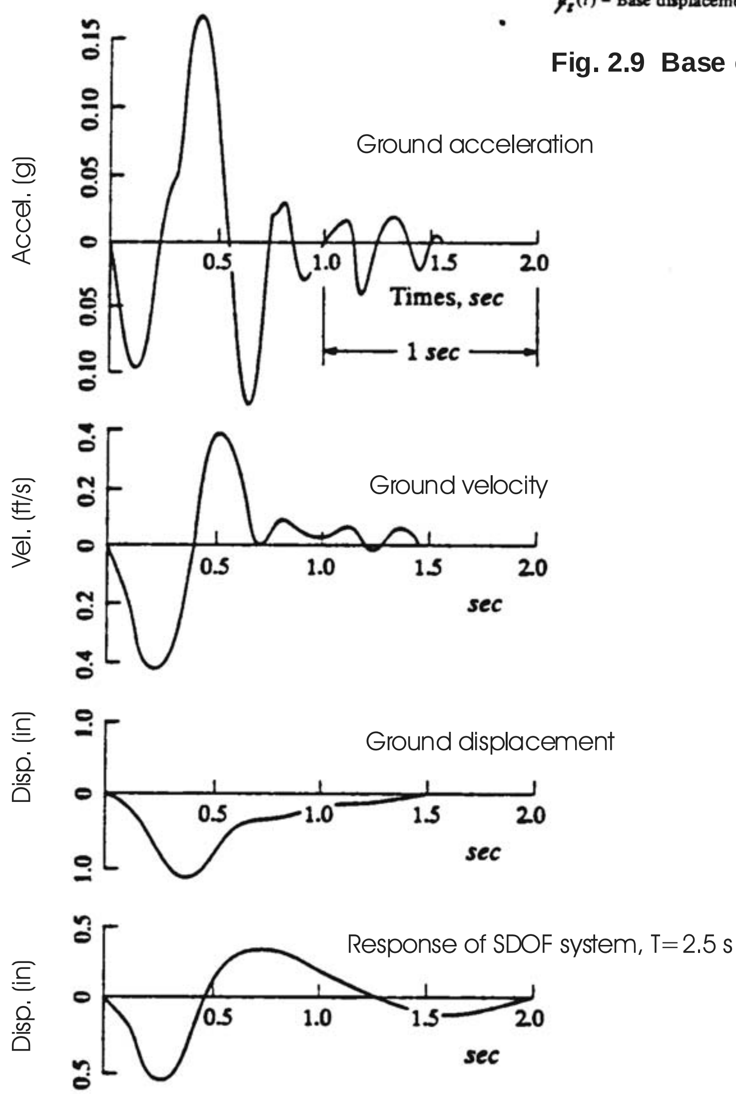


Fig. 2.10 Example of ground motion excitation (1957 Port Hueneme earthquake) and response of SDOF system with a 2.5 sec natural period.

Note: Figures 2.1 to 2.10 have been adapted from Chopra (2000)

3. EARTHQUAKE EXCITATION

3.1 EARTHQUAKE DAMAGE MECHANISMS

Earthquakes can damage structures in various ways, such as:

- by inertial forces generated by severe ground shaking;
- by direct fault displacement at the site;
- by foundation failure due to consolidation, settlement and liquefaction of the supporting soil;
- by landslides, or other surficial movements;
- by water waves generated by seismic motions (tsunamis & seiches);
- by fires resulting from earthquake shaking;
- by large-scale tectonic changes in ground elevation.

Earthquake ground motion is usually measured by strong-motion accelerographs, which record the acceleration of the ground at particular locations.

The recorded accelerograms are generally corrected for instrument errors and adjusted for baseline, and are integrated to obtain velocity and displacement time histories (see Fig. 3.1).

The peak values of ground acceleration, velocity and displacement are of most interest in seismic design. These parameters, in combination with other factors such as magnitude, epicentral distance, distance to the fault, duration of strong shaking, soil conditions of the site, and frequency content of the motion, affect the seismic behaviour of a structure.

3.2 CHARACTERISTICS OF EARTHQUAKE GROUND MOTION

The characteristics of earthquake ground motion which are of most interest in earthquake engineering applications are:

1. Peak ground motions (acceleration, velocity and displacement) primarily influence the vibration amplitudes
2. Duration of strong motion has a pronounced effect on the severity of the shaking.
3. Frequency content spectral shapes relate to frequencies or periods of vibration of a structure (resonance conditions).

A ground motion with moderate peak acceleration and a long duration may be more damaging than a ground motion with a larger acceleration and a shorter duration. In a structure, ground motion is amplified the most when the frequencies that dominate the motion are close to the vibration frequencies of the structure.

3.3 DEFINITION OF RESPONSE SPECTRUM

The earthquake response spectrum concept was introduced by Biot and Housner in the early 1940s. The response spectrum is a graphical representation of the maximum response of a single-degree-of-freedom oscillator at various frequencies or periods. The computational process to obtain the response spectrum for a prescribed ground motion and system damping is shown graphically in Fig. 3.2. The effect of varying the system damping is shown in Fig. 3.3.

In contrast with the Response Spectrum, the Fourier Spectrum (see Fig. 3.4) is a graphical representation of the frequency content of a recorded motion. The Fourier Spectrum provides and indication of which are the dominant frequencies of the Fourier harmonic components that make up the recorded, or computed, motion time history.

Several different definitions of response spectra are employed, but the following have come into fairly uniform use:

SD = $|u|_{max}$ = displacement response spectrum
SV = $|\underline{u}|_{max}$ = velocity response spectrum
SA = $|\ddot{u}+\ddot{y}_g|_{max}$ = acceleration response spectrum
PSV = p·SD = pseudo-velocity response spectrum
PSA = p^2·SD = pseudo-acceleration response spectrum

Examples of spectral values for SD, PSV and PSA are shown in Fig. 3.5. PSV and PSA are given the prefix "pseudo-" because they are not truly the peak values of velocity and acceleration, although they have the correct dimensions. For most situations in earthquake engineering, PSA≈SA, but PSV and SV are not necessarily the same in the same period range in which PSA and SA are nearly equal. As shown in Fig. 3.6, for systems with low frequency (high period) or high frequency (low period) SV and PSV can be significantly different.

Since SD, PSV and PSA are essentially a measure of the maximum relative displacement of the system, and in the range of structural periods of practical interest PSA≈SA and PSV≈SV, it is possible to present all this information in one single logarithmic plot, such as the one shown in Fig. 3.7. In this plot, the horizontal axis represents structural periods (or frequencies), the vertical axis gives a measure of the system velocity, and the diagonal axes give a measure of the system displacement and acceleration. Hence, the quantities of main interest in design can be readily obtained from a single response spectrum plot.

3.4 EARTHQUAKE DESIGN SPECTRUM

- The detailed characteristics of future earthquakes is not known.
- Generally, earthquake design spectra are obtained by averaging a set of response spectra from records with common characteristics.
- In practice, design spectra are presented as smooth curves of straight lines.

- Difference between ***Response Spectrum*** and ***Design Spectrum***:

  A **Response Spectrum** is a plot of the maximum response of a single-degree-of-freedom oscillator with different frequencies and damping ratios to a specific ground motion.

  A **Design Spectrum** is a specification of the seismic design force or displacement of a structure having a certain frequency or period of vibration and damping.

  These two concepts are illustrated in Fig. 3.8 where the response spectrum for recorded motions during the 1994 Northridge earthquake is compared with the NBCC spectra.

3.5 TIME HISTORY RESPONSE ANALYSIS

The response to earthquake loading can be obtained directly from Duhamel's (or Convolution) integral:

$$u(t) = \frac{1}{p_d}\int_0^t \ddot{y}_g(\tau)e^{-\zeta p(t-\tau)} \sin p_d(t-\tau)d\tau$$

- It is convenient to express the forces developed in the structure during the earthquake in terms of the effective inertia forces.

- The inertia force is the product of the mass and the total (absolute) acceleration. This total acceleration can be expressed as

$$\ddot{x}(t) = -\zeta p\dot{u}(t) - p^2 u(t)$$

- For small values of damping, the total acceleration can be approximated as

$$\ddot{x}(t) \approx -p^2 u(t)$$

- The effective earthquake force (base shear) is then given as

$$Q(t) = m\ddot{x}(t) \approx -mp^2 u(t)$$

- The overturning moment acting on the base of the structure can be determined by multiplying the inertia force by the story height (h):

$$M(t) = hm\ddot{x}(t) \approx -hmp^2 u(t)$$

3.6 RESPONSE SPECTRUM ANALYSIS

A time history analysis may require considerable computational effort, even for simple structural systems. For practical structural design only the maximum response quantities are required in most of the cases.

The maximum base shear and maximum overturning moment can then be computed using peak response values obtained from the response spectrum plot:

$$Q_{\max} = mp^2\,SD = m\,PSA = m\,SA$$

$$M_{\max} = hmp^2\,SD = hQ_{\max}$$

3.7 NONLINEAR ANALYSIS

During actual severe ground shaking buildings and other structures may be deformed beyond their elastic range. In the elastic range, the material is resilient like a spring and returns to its original condition without damage. Beyond the elastic range; deformations may enter the plastic range, where damage sets in.

The equation of motion of a system specifies the requirement of equilibrium between the applied force, the inertia force, the damping force, and the spring force. The last three forces depend on the physical properties of the system (mass, damping and stiffness). If these properties do not vary with time, the system is said to be linear and the solution methods discussed above are applicable. However, if any of these physical characteristics vary with time, the system becomes nonlinear and special methods must be devised for its solution. In general, the mass is time invariant. Since damping characteristics cannot, in any case, be defined with certainty, it is not unreasonable to assume that they also remain constant with time.

The response of real structures, when subjected to large dynamic loads, often involves significant nonlinear behaviour. In general, nonlinear behaviour includes the effects of large displacements and or nonlinear material properties. The more common type of nonlinear behaviour is when the material stress-strain, or force-deformation, relationship is nonlinear. This is because of the modern design philosophy that ***a well-designed structure should have a limited number of members which require ductility and that the failure mechanism be clearly defined*** (Wilson, 1996). Such an approach minimizes the cost of repair after a major earthquake.

A large number of very practical structures, when subjected to static or dynamic loading, have a limited number of points or members in which nonlinear behaviour takes place. Local buckling of diagonals, uplifting at the foundation, contact between different parts of the structures and yielding of a few elements are examples of such systems. For dynamic loads, it is becoming common practice to add concentrated damping, base isolation and other energy dissipation elements.

The global dynamic equilibrium equation, at time *t,* of an elastic structure with nonlinear or energy dissipating elements can be written in the following form:

$$\mathbf{m\ddot{u}(t) + c\underline{u}(t) + R(t) = -m\ddot{y}_g(t)}$$

In this equation, R(t) is the restoring force due to the sum of the forces in the linear and nonlinear elements of the system. For a linear, elastic system, $R(t) = ku(t)$. There are several efficient techniques for evaluating the response of nonlinear structural systems, but in general, a nonlinear dynamic analysis is a very computationally demanding task.

Key Issues in Nonlinear analysis:

- The basic aim is to make better assumptions about stiffnesses, in order to get more accurate displacements, deformations and forces.
- Modelling decisions become more complex since stiffnesses can vary in complex ways, and be dependent on many parameters.
- Numerical computations are more expensive.
- Cannot superimpose results for separate load cases.
- Some basic and difficult questions:
  (1) how much sophistication is justified?
  (2) how much accuracy can be expected?
  (3) what are the weak links in the chain of assumptions?
  (4) how sensitive are the results to changes in specific assumptions?
- Nonlinear analysis results must be used with even more judgment than those from linear elastic analyses.

Is nonlinear analysis needed?

- Linear analysis can be used to estimate strength demand. Nonlinear analysis can be used to estimate deformation demand. Strength demand is reasonable for design, while deformation demand is more reasonable for damage prediction.

- Results must be interpreted cautiously since there are many uncertainties in modelling and load selection. Ideally the uncertainties should be quantified, and their effects accounted for, which in practice this is not yet feasible.

- Nonlinear analysis is valuable for:
  a) research, including post-earthquake evaluation.
  b) studies of expensive or critical structures, where thorough analyses can be performed.
  c) design where the nonlinearities are isolated and controlled (e.g. base isolation).
  d) retrofit of important existing structures.

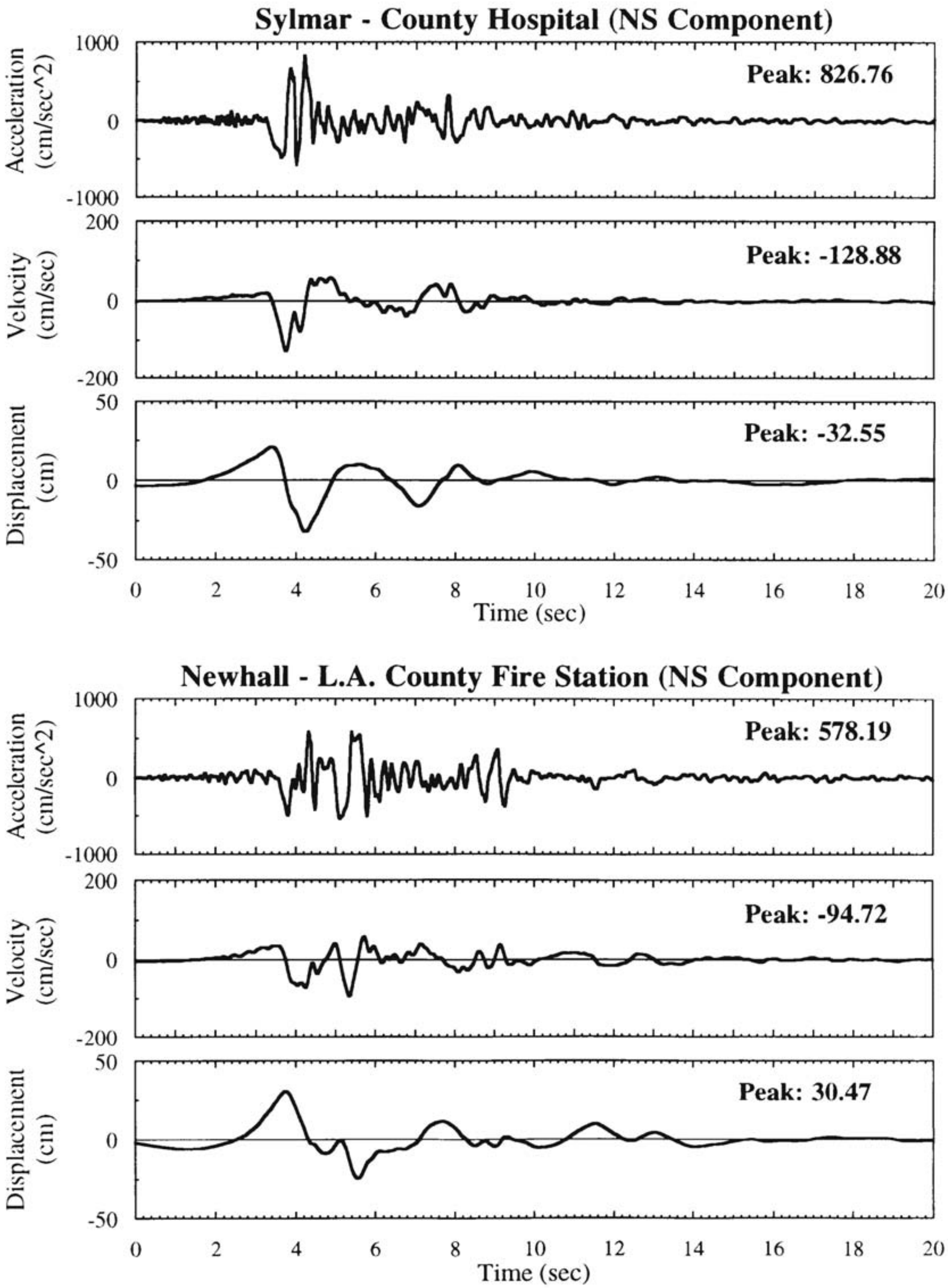


Fig. 3.1 Recorded ground motions during the 1994 Northridge earthquake.

Deformation response spectrum

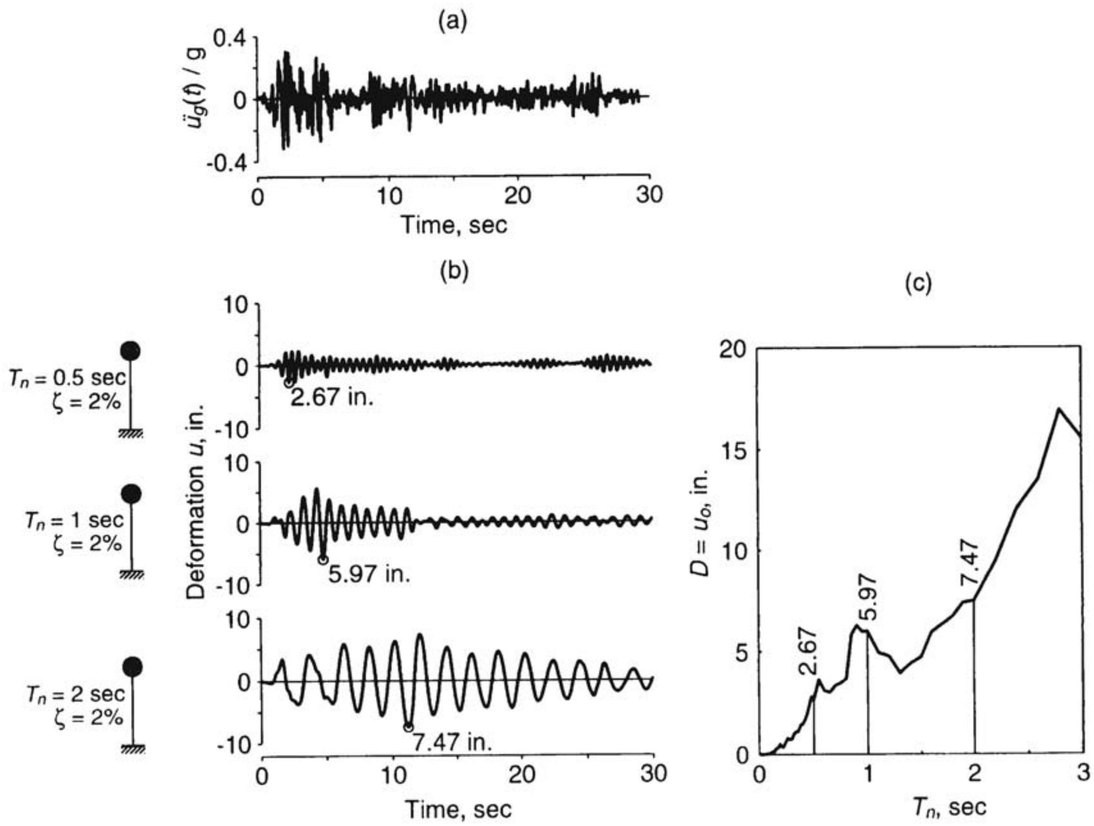


Fig. 3.2 Computation of deformation response spectrum (after Chopra, 1995).

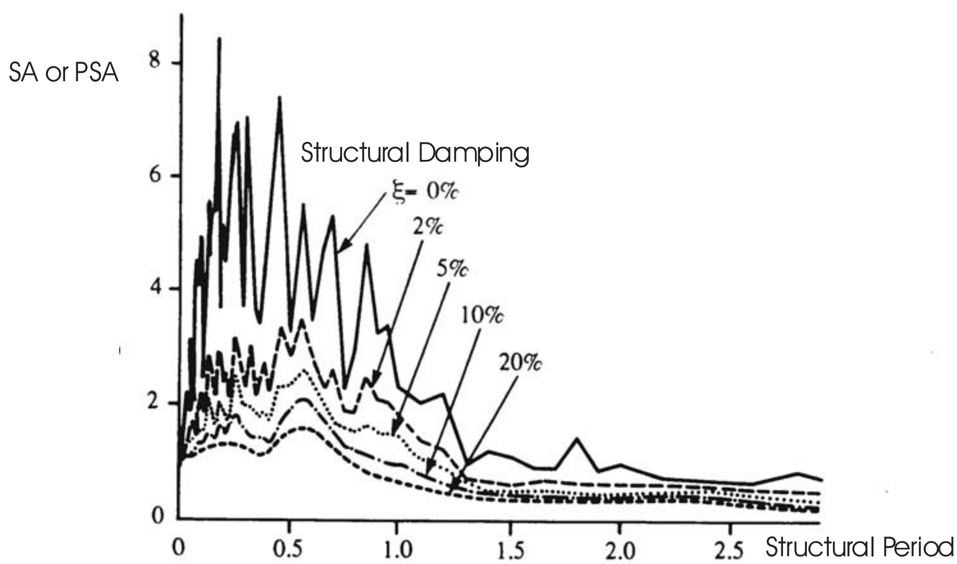
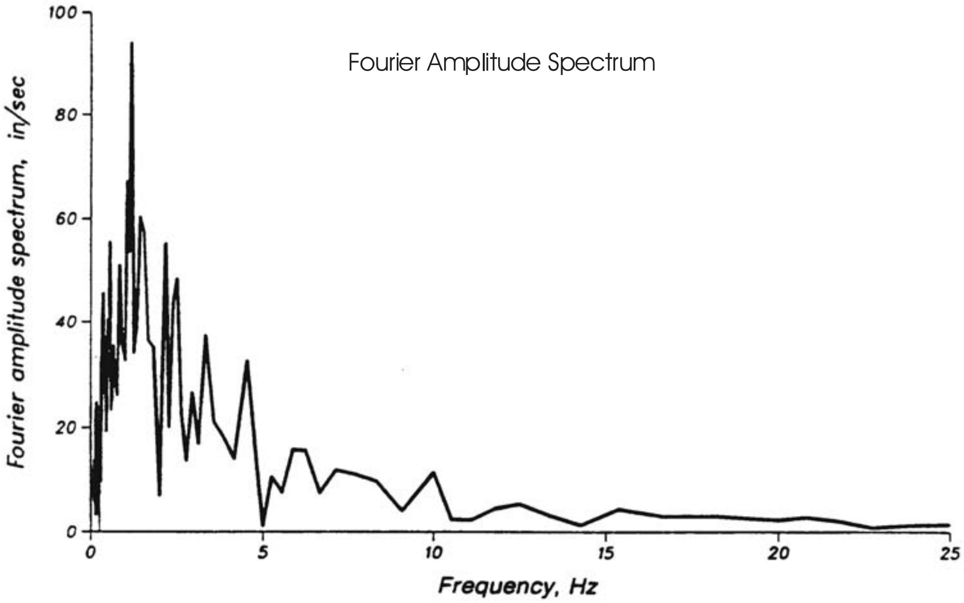


Fig. 3.3 Information contained in a response spectrum plot.



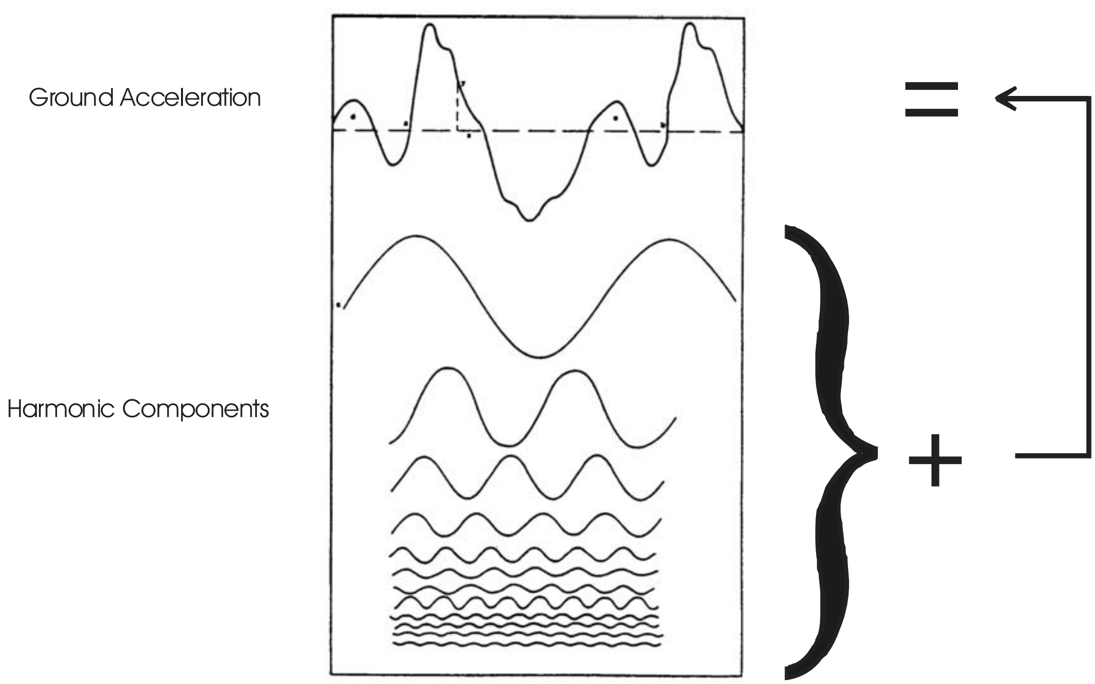


Fig. 3.4 Fourier spectrum and frequency content of earthquake ground motion (1940 El Centro).

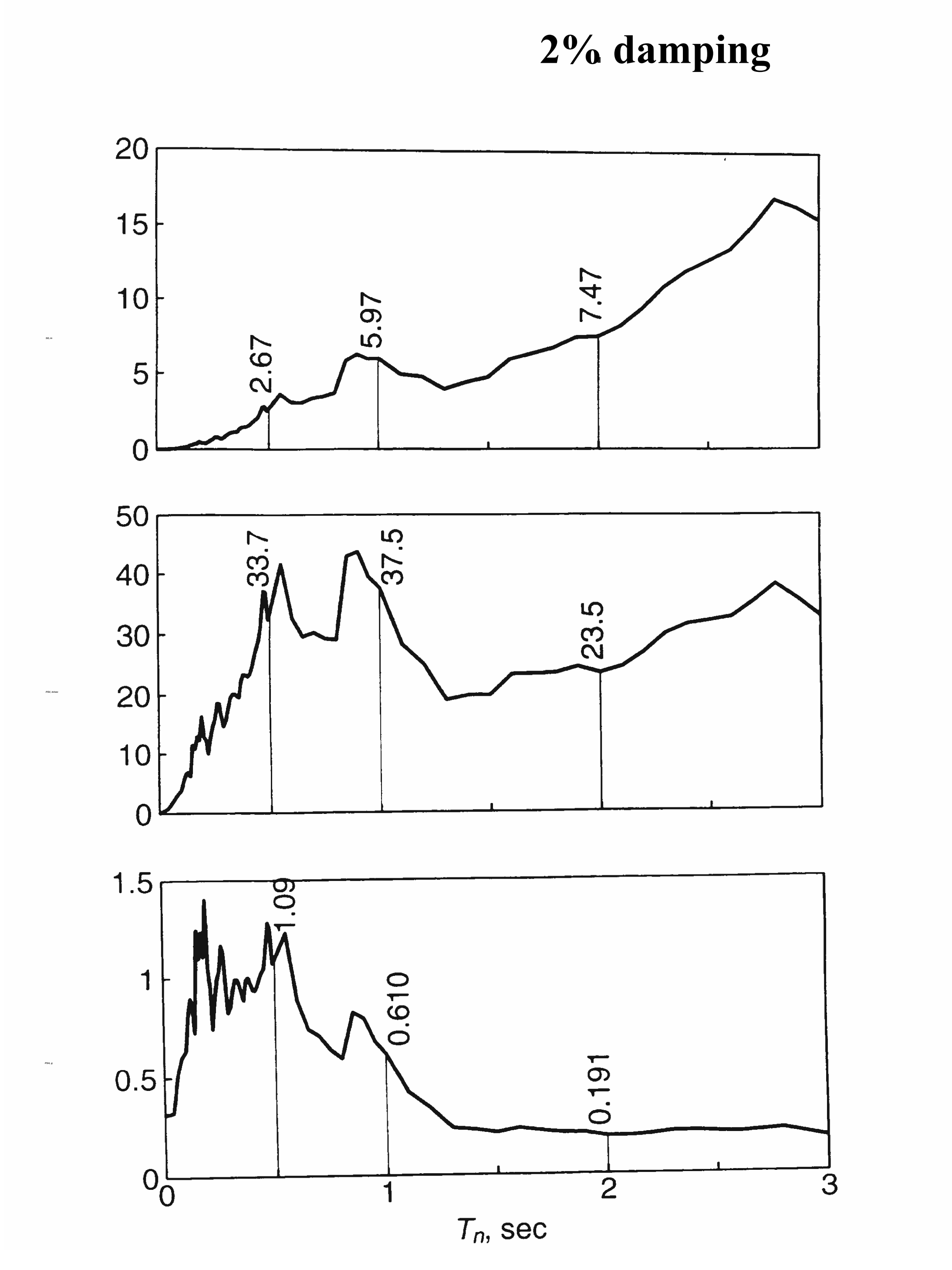
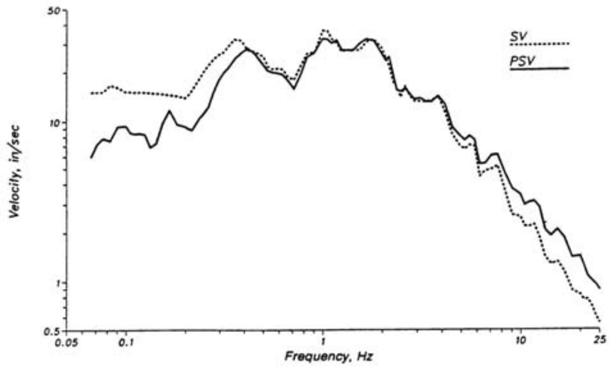


Fig. 3.5 Spectral Responses of Engineering Interest (after Chopra, 1995).



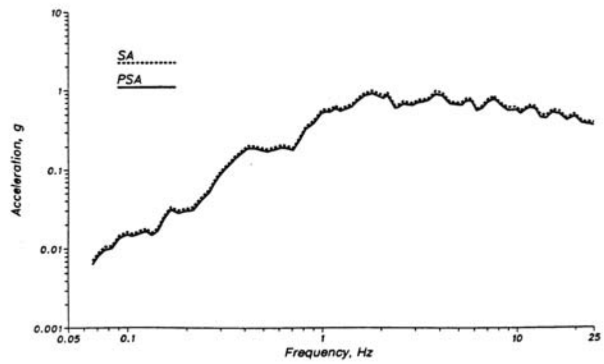


Fig. 3.6 Comparison of SV vs PSV and SA vs PSA response spectra for systems with 5% damping (El Centro 1940 record).

Combined SD-PSV-PSA response spectrum

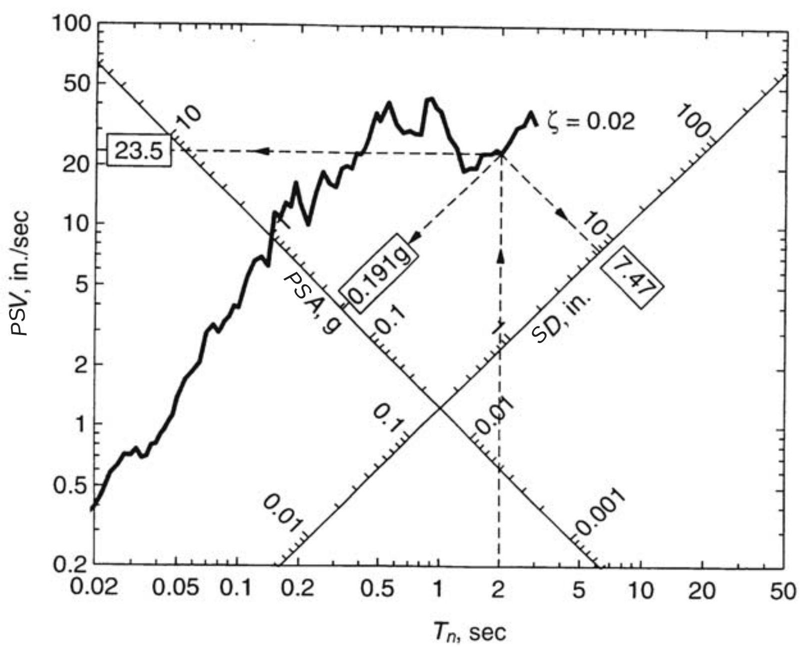


Fig. 3.7 Tri-partite Log-Log plot of response spectrum (after Chopra, 1995).

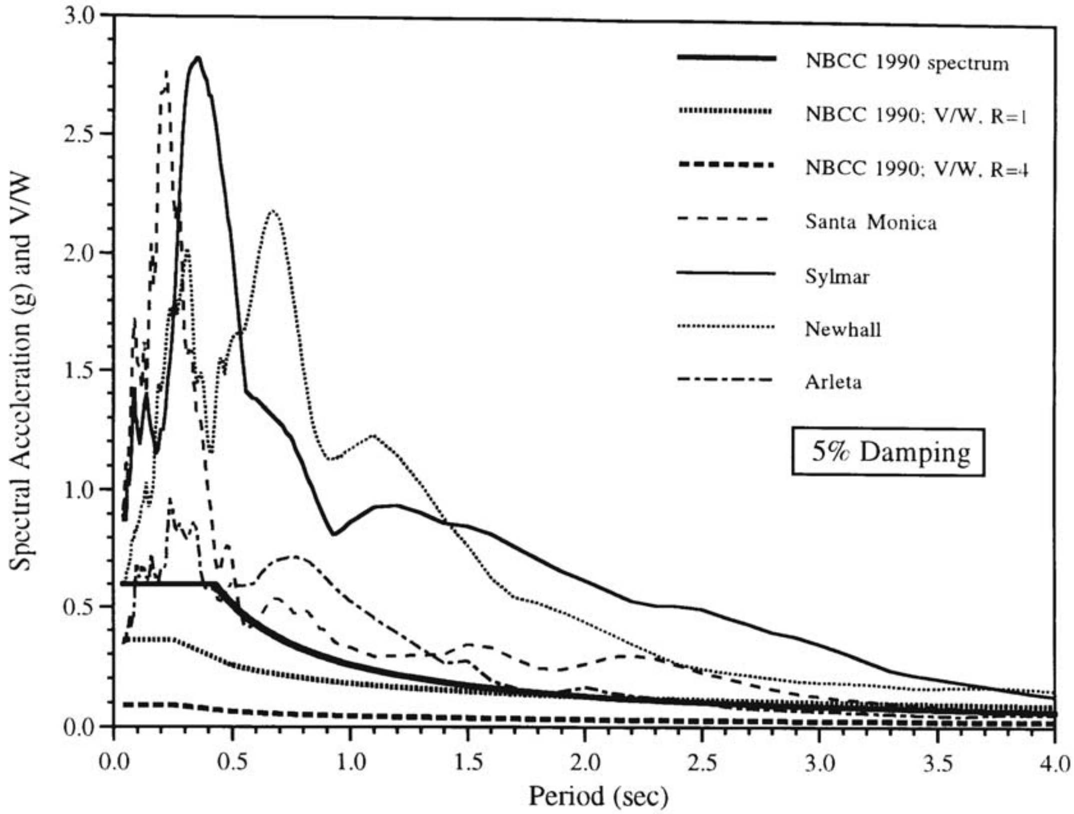


Fig. 3.8 Comparison of response spectra from 1994 Northridge earthquake ground motions and NBCC design spectra.

4. MULTI-DEGREE-OF-FREEDOM (MDOF) SYSTEMS

4.1 FORMULATION OF EQUATIONS OF MOTION

Structures with several layers of mass (such as buildings with several floors) do not behave like SDOF systems. Such structures must be evaluated as MDOF systems whose vibrations will be a combination of the vibration of each layer.

In the dynamic analysis of typical buildings the mass of the structure is assumed to be concentrated at the floor levels and to be subjected to lateral displacements only.

For the three-storey building shown below, its dynamic behaviour is completely defined by the three storey displacements x_a, x_b, and x_c.

The equation of motion of any storey is the equation of dynamic equilibrium of all the forces acting on the storey mass:

$$\begin{aligned} \mathbf{F_{Ia}(t) + F_{Da}(t) + F_{Sa}(t) = P_a(t)} \\ \mathbf{F_{Ib}(t) + F_{Db}(t) + F_{Sb}(t) = P_b(t)} \\ \mathbf{F_{Ic}(t) + F_{Dc}(t) + F_{Sc}(t) = P_c(t)} \end{aligned}$$

For this lumped mass system, the inertia forces are given simply as the product of the storey mass and the storey acceleration:

$$\mathbf{F_{Ia}(t) = M_a\ddot{x}_a(t)} \qquad \mathbf{F_{Ib}(t) = M_b\ddot{x}_b(t)} \qquad \mathbf{F_{Ic}(t) = M_c\ddot{x}_c(t)}$$

which may be represented in matrix form as

$$\begin{Bmatrix} F_{Ia}(t) \\ F_{Ib}(t) \\ F_{Ic}(t) \end{Bmatrix} = \begin{bmatrix} M_a & 0 & 0 \\ 0 & M_b & 0 \\ 0 & 0 & M_c \end{bmatrix} * \begin{Bmatrix} \ddot{x}_a(t) \\ \ddot{x}_b(t) \\ \ddot{x}_c(t) \end{Bmatrix}$$

In a symbolic form, this expression may written as $\mathbf{\{F_I\} = [M]\{\ddot{x}\}}$ in which $\{F_I\}$ is the inertia force vector, $\{\ddot{x}\}$ is the acceleration vector, and [M] is the mass matrix.

The elastic forces F_s depend on the displacement of the system and may be expressed in terms of the stiffness influence coefficients:

$$\begin{aligned} \mathbf{F_{Sa}(t) = k_{aa}x_a(t) + k_{ab}x_b(t) + k_{ac}x_c(t)} \\ \mathbf{F_{Sb}(t) = k_{ba}x_a(t) + k_{bb}x_b(t) + k_{bc}x_c(t)} \\ \mathbf{F_{Sc}(t) = k_{ca}x_a(t) + k_{cb}x_b(t) + k_{cc}x_c(t)} \end{aligned}$$

The general stiffness influence coefficient k_{ij} may be defined as the force corresponding to displacement coordinate *i* resulting from a unit displacement of coordinate *j*. In matrix form this expression can be written as

$$\begin{Bmatrix} F_{Sa}(t) \\ F_{Sb}(t) \\ F_{Sc}(t) \end{Bmatrix} = \begin{bmatrix} k_{aa} & k_{ab} & k_{ac} \\ k_{ba} & k_{bb} & k_{bc} \\ k_{ca} & k_{cb} & k_{cc} \end{bmatrix} * \begin{Bmatrix} x_a(t) \\ x_b(t) \\ x_c(t) \end{Bmatrix}$$

In a symbolic form, this expression may written as **$\{F_S\} = [K]\{x\}$** in which $\{F_s\}$ is the elastic force vector, {x} is the displacement vector, and [K] is the stiffness matrix of the structure.

By analogy with the expression for elastic forces and if the damping is assumed to be of the viscous type, the damping forces could be written as **$\{F_D\} = [C]\{\underline{x}\}$** in which $\{F_D\}$ is the damping force vector, $\{\underline{x}\}$ is the velocity vector, and [C] is the damping matrix.

The equation of equilibrium can then be written as

$$[M]\{\ddot{x}\} + [C]\{\underline{x}\} + [K]\{x\} = \{P(t)\}$$

4.2 VIBRATION MODE SHAPES AND FREQUENCIES

The dynamic response of a structure depends upon two important factors:

1) its period of vibration T or frequency p,
2) its assumed displacement shape

The free vibration behaviour of a structure is expressed by the equations of motion adapted to the special condition of no damping ([C]=[0]) and with no applied loading ({P(t)}={0}). In this case

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

The motions of the system in free vibration are simple harmonic, which may be written as **$\{x\} = \{A\}\sin(pt)$**. Substituting this expression and its corresponding second order derivative, the equations of motion become

$$-p^2[M]\{A\} + [K]\{A\} = \{0\}$$

or

$$[K]\{A\} = p^2[M]\{A\}$$

The solution of this eigenvalue equation for a system having **N** degrees of freedom provides a vibration frequency $\mathbf{p_n}$ (or period $\mathbf{T_n=2\pi/p_n}$) and a vibration shape (*Mode Shape)* $\{\varphi_n\}$ for each of its N modes of vibration.

The vibration mode shapes $\{\varphi_n\}$ of any MDOF system have two orthogonality properties that make possible an important simplification in the general equations of motion:

$$\{\varphi_n\}^T[M]\{\varphi_m\} = 0 \text{ for } m \neq n$$
$$\{\varphi_n\}^T[K]\{\varphi_m\} = 0 \text{ for } m \neq n$$

An illustrative example of computation of mode shapes and natural frequencies for a 5-DOF system with different combinations of floor stiffnesses and masses is presented in the Appendix.

4.3 MODAL SUPERPOSITION METHOD OF ANALYSIS

There are N independent mode shapes for an N-degree-of-freedom undamped system. Any arbitrary displaced shape of the structure may be expressed in terms of the amplitudes of these shapes. In general

$$x_i(t) = \sum_{n=1}^{N} \phi_{ni} Y_n$$

where $\mathbf{Y_n}$ is the amplitude of the *n*th mode. The set of values Y_n expressed in vector form **{Y}** is called the *normal* coordinates of the system. The equations of motion can be expressed in terms of the normal coordinates as follows

$$\mathbf{[M][\Phi]\{\ddot{Y}\} + [C][\Phi]\{\underline{Y}\} + [K][\Phi]\{Y\} = \{P(t)\}}$$

Making use of the orthogonality properties, these equations can be converted into a set of N independent second order differential equations similar to those used to define the dynamic behaviour of SDOF systems:

$$\mathbf{\ddot{Y}_n(t) + 2\zeta_n p_n \underline{Y}_n(t) + p_n^2 Y_n(t) = P_n^*(t)/M_n^*}$$

where

$\mathbf{M_n^* = \{\varphi_n\}^T[M]\{\varphi_n\}}$ is the *Generalized mass* of the *n*th mode shape

$\mathbf{C_n^* = \{\varphi_n\}^T[C]\{\varphi_n\} = 2\zeta_n p_n M_n^*}$ is the *Generalized damping*

$\mathbf{K_n^* = \{\varphi_n\}^T[K]\{\varphi_n\} = p_n^2 M_n^*}$ is the *Generalized stiffness*

$\mathbf{P_n^*(t) = \{\varphi_n\}^T\{P(t)\}}$ is the *Generalized Loading*

The normal coordinates (mode shapes) of a MDOF system reduce its equations of motion to a set of independent equations, one for each mode of vibration.

The dynamic response of the MDOF is then computed by superposition of individual modal components obtained from the solution of the above equation ***(Modal Superposition Method)***

The steps for computing the dynamic response of a MDOF system by modal analysis are:

1. Define the structural properties: mass matrix, stiffness matrix, damping matrix (or modal damping ratios).
2. Determine the natural frequencies and mode shapes.
3. Compute modal responses of interest.
4. Combine the contributions of all modes to determine the total response.

4.4 EARTHQUAKE RESPONSE ANALYSIS

For a ground excited MDOF system, the uncoupled differential equations of motion can be easily adapted to compute the relative motion of the structure with respect to its moving base by calculating first the effective floor loads,

$$\mathbf{Pi_{eff}(t) = -M_i\ddot{y}_g(t)}$$

and the complete effective applied load vector is given by

$$\mathbf{\{P_{eff}(t)\} = [M]\{-1\}\ddot{y}_g(t)}$$

where {-1} represents a vector of negative ones. The ***Generalized effective earthquake loading*** for mode *n* is given by

$$\mathbf{P^*_{eff}(t) = \{\varphi_n\}^T[M]\{-1\}\ddot{y}_g(t) = \Gamma_n\ddot{y}_g(t)}$$

in which $\mathbf{\Gamma_n= \{\varphi_n\}^T[M]\{-1\}}$ represents the earthquake modal participation factor for mode *n*.

The uncoupled equations of motion are therefore:

$$\mathbf{\ddot{Y}_n(t) + 2\zeta_n p_n \underline{Y}_n(t) + p_n^2 Y_n(t) = \Gamma_n\ddot{y}_g(t)/M_n^*}$$

The response of the *n*th mode at any time *t* may be obtained by numerical evaluation of Duhamel's integral:

$$Y_n(t) = \frac{\Gamma_n}{p_{d_n} M_n^*}\int_0^t \ddot{y}_g(\tau)e^{-\zeta_n p_n(t-\tau)}\sin p_{d_n}(t-\tau)d\tau$$

The complete displacement of the structure at time *t* is then obtained by superposition of all the modal responses:

$$\{x(t)\} = \sum_{n=1}^{N}\{\phi_n\}Y_n(t)$$

An example of modal contributions to shear forces in a frame building is shown in Fig. 4.1.

4.5 RESPONSE SPECTRUM ANALYSIS

The entire time history of displacements and forces can be computed using the method of superposition described above.

The uncoupled equations of motion are equivalent to those for SDOF systems. Thus, the maximum response of each mode of a MDOF system can be obtained from the earthquake response spectrum method described for SDOF systems.

The spectral displacement, SD_n obtained from a response spectrum for a system with period T_n and damping ζ_n can be used to obtain the maximum response of mode *n*:

$$Y_{n_{\max}} = \frac{\Gamma_n}{M_n^*} SD_n$$

The distribution of maximum modal displacements is given by

$$\{x_{n_{\max}}\} = \{\varphi_n\} Y_{n_{\max}} = \{\varphi_n\} \frac{\Gamma_n}{M_n^*} SD_n$$

Similarly, the distribution of maximum effective earthquake forces in this mode becomes

$$\{q_{n_{\max}}\} = [M]\{\varphi_n\} p_n^2 Y_{n_{\max}} = [M]\{\varphi_n\} \frac{\Gamma_n}{M_n^*} SA_n$$

The maximum base shear for this mode is given by

$$Q_{n_{\max}} = \frac{\Gamma_n^2}{M_n^*} SA_n$$

Examples of modal base shears for a 10-storey frame building obtained from a response spectrum analysis are presented in Fig. 4.2.

Since the individual maximum modal responses may not occur at the same time, the total maximum values cannot not, in general, be obtained as the sum of the individual modal maxima. An approximation to the total maximum response may be obtained by proper combination of modal maxima (sum of absolute values, SRSS, CQC, etc.)

4.6 MODAL COMBINATIONS

From a response spectrum the maximum modal deformations and member forces can be readily computed. These can be considered as "correct" or "exact" values. Since modal maxima do not occur at the same time, in general, any combination of modal maxima may lead to results that may be either conservative or unconservative, depending on what modal combination technique is being used and on the dynamic properties of the system being analysed. Three of the most commonly used modal combination methods are:

a) Sum of the absolute values: This method leads to very conservative results. It assumes that maximum modal values occur at the same time. The response of any given degree of freedom of the system is estimated as a linear combination of the absolute values of the maximum modal responses, i.e.;

$$x_{i_{max}} = \sum_{n=1}^{L} \left| x_{i,n_{max}} \right|$$

in which i is the degree of freedom of interest and L is the number of modes being considered in the analysis. Member forces are computed using a similar expression.

b) Square root of the sum of the squares (SRSS): This method assumes that the individual modal maxima are statistically independent. The response of any given degree of freedom of the system is estimated as

$$x_{i_{max}} = \sqrt{\sum_{n=1}^{L} (x_{i,n_{max}})^2}$$

The SRSS method generally leads to values that are closer to the "exact" ones than those obtained using the sum of the absolute values. The results, however, could be either on the conservative or unconservative side. When the modal frequencies of the system are closely spaced, the results from an SRSS analysis can be significantly unconservative.

c) Complete quadratic combination (CQC): This method was introduced in the early 1980's and it is based on random vibration theory. It has found wide acceptance in structural dynamics and has been incorporated in several commercial analysis programs. The following double summation is used to estimate maximum responses,

$$x_{i_{max}} = \sqrt{\sum_{n=1}^{L} \sum_{m=1}^{L} x_{i,n_{max}} \rho_{n,m} x_{i,m_{max}}}$$

In which, $\boldsymbol{\rho}$ is a cross-modal coefficient (always positive), which for constant damping is evaluated by

$$\rho_{\mathbf{n,m}} = \frac{8\zeta^2(1+\mathbf{r})\mathbf{r}^{1.5}}{(1-\mathbf{r}^2)+4\zeta^2\mathbf{r}(1+\mathbf{r}^2)}$$

where $\mathbf{r} = \mathbf{p_n} / \mathbf{p_m}$ and must be equal to or less than 1.0.

Similar equations can be applied for the computation of member forces, interstorey deformations, base shears and overturning moments.

4.7 PARTICIPATING MASS

The number of modes to be included in the computation of the responses of a MDOF system can be determined by reference to the participating mass. Building codes generally require that a least 90% of the participating mass be included in the calculation of response for each principal horizontal direction. Such requirement is based on a *unit base acceleration* and the corresponding base shear due to that load.

The base shear for an undamped system in which SA = 1g for all modes, is therefore given as

$$\mathbf{V_{max}} = \sum_{\mathbf{n}=1}^{\mathbf{N}} \mathbf{Q_{max}} = \sum_{\mathbf{n}=1}^{\mathbf{N}} \frac{\mathbf{\Gamma_n^2}}{\mathbf{M_n^*}}\mathbf{g}$$

It can be readily demonstrated that when all the modes are accounted for, the base shear should be equal to ***m***g, where ***m*** is the total mass of the system. The base shear computed including only "L" modes, will be

$$\mathbf{V_{max}} \approx \sum_{\mathbf{n}=1}^{\mathbf{L}} \mathbf{Q_{max}} = \sum_{\mathbf{n}=1}^{\mathbf{L}} \frac{\mathbf{\Gamma_n^2}}{\mathbf{M_n^*}}\mathbf{g}$$

A modal mass participation factor, MPF, can now be defined in terms of the of the total mass, **m**, by

$$\mathbf{MPF_n} = \frac{1}{\mathbf{m}}\frac{\mathbf{\Gamma_n^2}}{\mathbf{M_n^*}}$$

If all the modes are included in the analysis, the sum of all the MPF's will be equal to 1.0. But it should be kept in mind that this MPF is based on the accuracy of the solution for a base motion only. It should not be used for other types of loadings acting on the structure.

4.8 RAYLEIGH'S METHOD

This procedure was developed by Lord Rayleigh to analyse vibrating systems using the law of conservation of energy.

Its principal use is for determining an accurate approximation of the natural frequency of a structure.

If the assumed vibration shape of a MDOF system is represented by a dimensionless shape vector **{ψ}**, the displacement vector may be expressed as

$$\{x(t)\} = \{\psi\}Y(t)$$

For an undamped system, the maximum kinetic energy of the structure is given by

$$KE_{max} = \tfrac{1}{2}\, p^2 Y^2 \{\psi\}^T [M]\{\psi\}$$

and the maximum potential energy is given by

$$PE_{max} = \tfrac{1}{2}\, Y^2 \{\psi\}^T [K]\{\psi\}$$

According to the principle of conservation of energy for undamped systems KE = PE, which leads to

$$p = \sqrt{\frac{\{\psi\}^T[K]\{\psi\}}{\{\psi\}^T[M]\{\psi\}}}$$

4.9 DAMPING

When the cyclic excitation on a structure ceases, its response tends to die away. This is the phenomenon known as damping. If the damping is assumed to be "viscous", i.e., the damping force varies as the velocity of the system relative to the ground, the mathematics become reasonably easy to solve. For this reason, the assumption of viscous damping is often adopted in analysis, although practical mechanisms of damping in buildings often follow somewhat different patterns.

Damping in building arises from a variety of causes, including aerodynamic drag, friction in connections and cladding, soil-structure interaction, and bond slip and cracking in reinforced concrete. These causes predominate when stresses are generally below yield. Plastic yielding gives rise to a different source of energy dissipation - hysteretic damping.

For the case of systems with equivalent viscous damping, the triple matrix product $[\varphi]^T[C][\varphi]$ may result in a matrix being or not diagonal, depending on the distribution of camping in the system. If the result is a diagonal matrix, then the system is said to be a "classically damped system" and the classical modal superposition method of analysis is applicable. For the case in which the product is a non-diagonal matrix, the system is said to be a "non-classically damped system". Such systems are not amenable to classical modal analysis, and they do not possess the same natural modes as the undamped system.

Classical damping is an appropriate idealization if damping mechanisms are distributed throughout the structure. Two common methods for constructing the damping matrix [C] are Rayleigh damping and the Caughey damping.

In the Rayleigh damping, [C] is constructed in terms of the mass and stiffness matrices as:

$$\mathbf{[C] = \alpha[M] + \beta[K]}$$

and the damping ratio for the *n*th mode is:

$$\boldsymbol{\zeta_n = \tfrac{1}{2}\,(\alpha/p_n + \beta\, p_n)}$$

The coefficients α and β can be determined from specified damping ratio for two distinct modes ("i" and "j" modes) as:

$$\frac{1}{2}\begin{bmatrix} \frac{1}{p_i} & p_i \\ \frac{1}{p_j} & p_j \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \begin{Bmatrix} \zeta_i \\ \zeta_j \end{Bmatrix}$$

In this approach, no more than two modes can have the same damping factor (see Fig. 4.3). If damping ratios are specified for more than two modes, Caughey damping allows the construction of a classical damping matrix. In this case:

$$[C] = [M]\sum_{r=0}^{N-1} \alpha_r ([M]^{-1}[K])^r$$

where N is the number of degrees of freedom in the system and α_r are constants. The damping ratio for the *n*th mode is given by

$$\zeta_n = \frac{1}{2}\sum_{r=0}^{N-1} \alpha_r p_n^{2r-1}$$

The coefficients α_r can be determined from the damping ratios specified in any number of modes and solving the algebraic system of equations resulting from the repeated application of the above equation. With α_r determined, then the damping matrix can be assembled.

A more general form for constructing the damping matrix is by making use of the following equation:

$$[C] = [M]\ \left(\sum_{r=0}^{N-1} \frac{2\zeta_r p_r}{M_r^*} \{\phi_r\}\{\phi_r\}^T\right)\ [M]$$

4.10 DISCUSSION

Most code loads are based on studies of building response using modal analyses in conjunction with the spectra from real earthquakes, or alternately in conjunction with a smoothed design spectrum.

The design spectrum in the design codes (see example in Fig. 4.4), is a simplified spectrum shape. It is used in dynamic analysis to give a better distribution of forces throughout the height of the structure, but the results are generally scaled to match the static code base shear. Nevertheless, it is a reasonable design spectrum. Figure 4.5 show the shear force vs. height for a uniform wall structure for two different first mode periods, T=1 and T=4 seconds. The weight and height of the structure is taken to be unity and the model is a 20 node vertical cantilever with equal masses at each node. A wall structure, as opposed to a frame, has been chosen because the higher modes have higher modal mass ratios, and the periods are more widely separated. This result in much larger second and third mode forces, especially for the longer period structures where the first mode spectral acceleration is quite small, but the higher modes may be in the region of peak spectral acceleration. The root-sum-square (RSS) result shown in the figures is for the first 3 modes. Figure 4.6 shows the moment distribution up the height for the same walls.

Note that for the T=1 second structure the total shear distribution, as given by the RSS results, is essentially given by the first mode shear. But for the T=4 second structure the second mode shear dominates. The bending response for both structures is essentially all from the first mode, as seen in Fig. 4.6. That the second mode can dominate the shear forces, while the first mode dominates the moment, is the cause for some of the adjustment factors for shear and moment distributions that will be mentioned later in the section on code forces. Frame structures behave differently in that the first mode dominates both the shear and moment distribution.

The examples shown have used the root-sum-square (RSS) method of combining modal forces. If the period difference between modes is small this method can be in error, and a more complicated method such as the complete-quadratic-combination (CQC) method is recommended. Many programs offer both types of combination.

One of the disadvantages of the modal method, employing a combination rule such as RSS, is that the sign of the forces is lost. Thus if you are considering axial load in a column you cannot tell if it is tension or compression. In addition equilibrium of maximum member forces, at say joints, will not be satisfied as the maximums may come at different times in the various members.

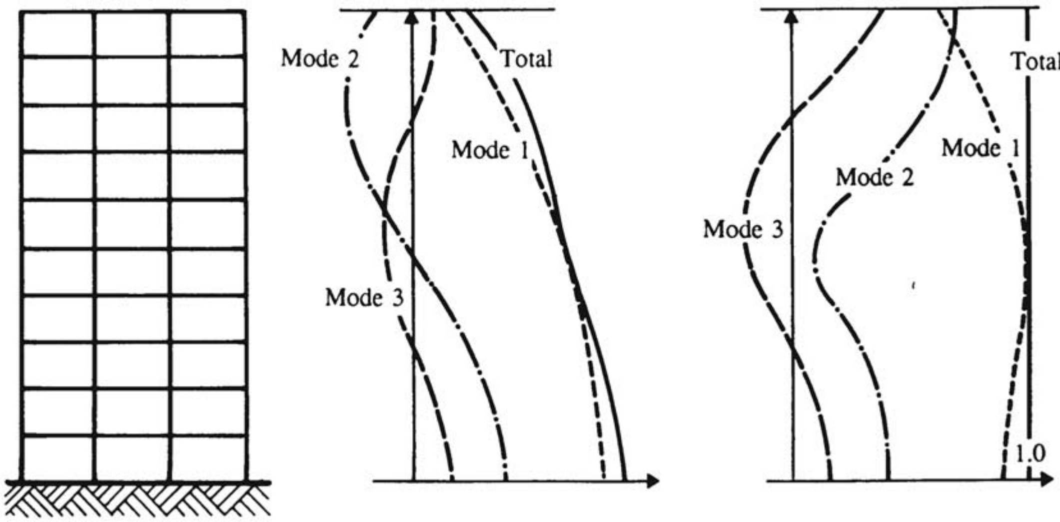


Fig. 4.1 Modal contributions to shear forces in a frame building.

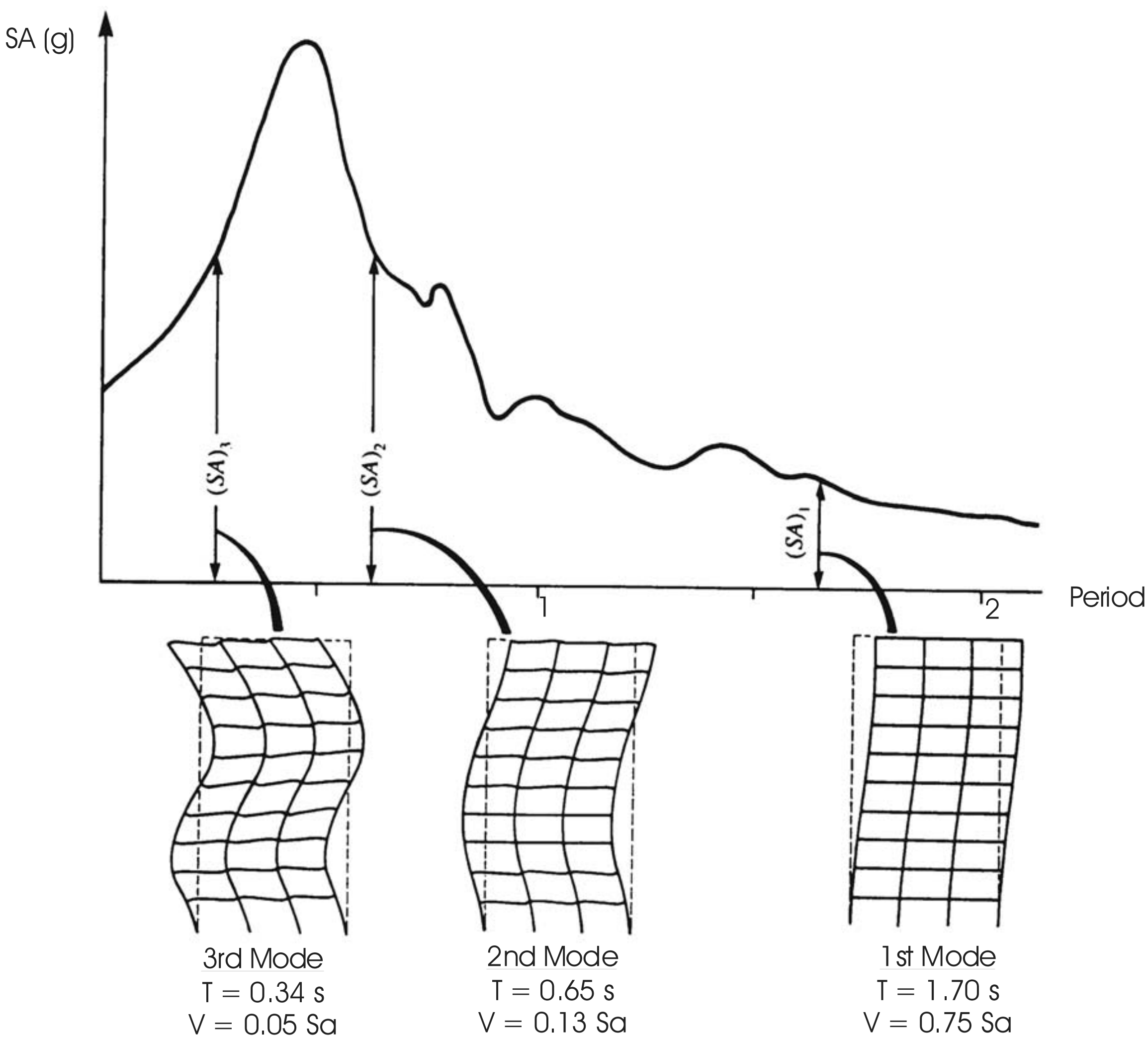


Fig. 4.2 Modal responses of a ten-storey frame building.

Rayleigh Damping: [C] = "[M] + $[K]

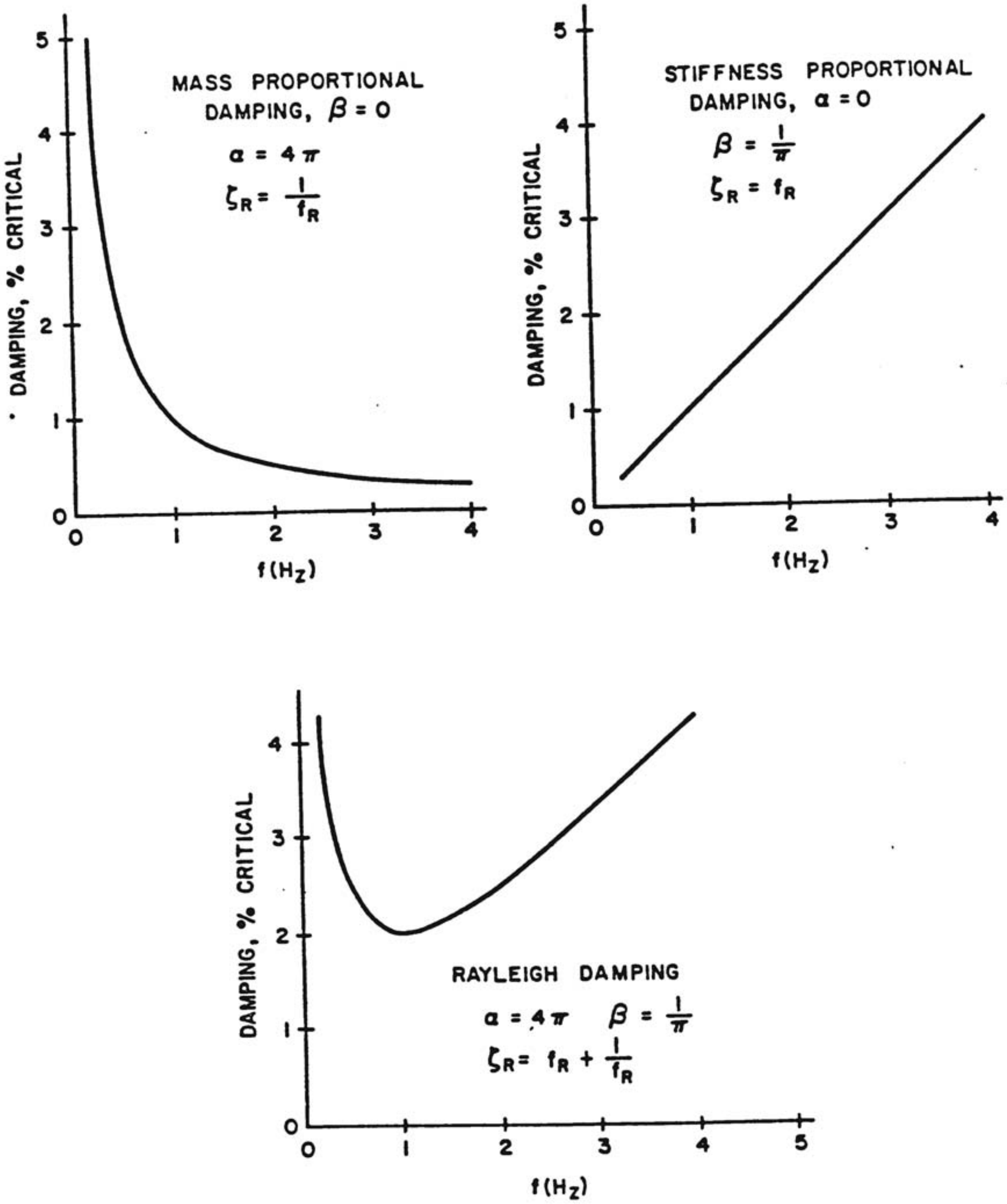


Fig. 4.3 Illustration of how Rayleigh damping affects modal damping ratios.

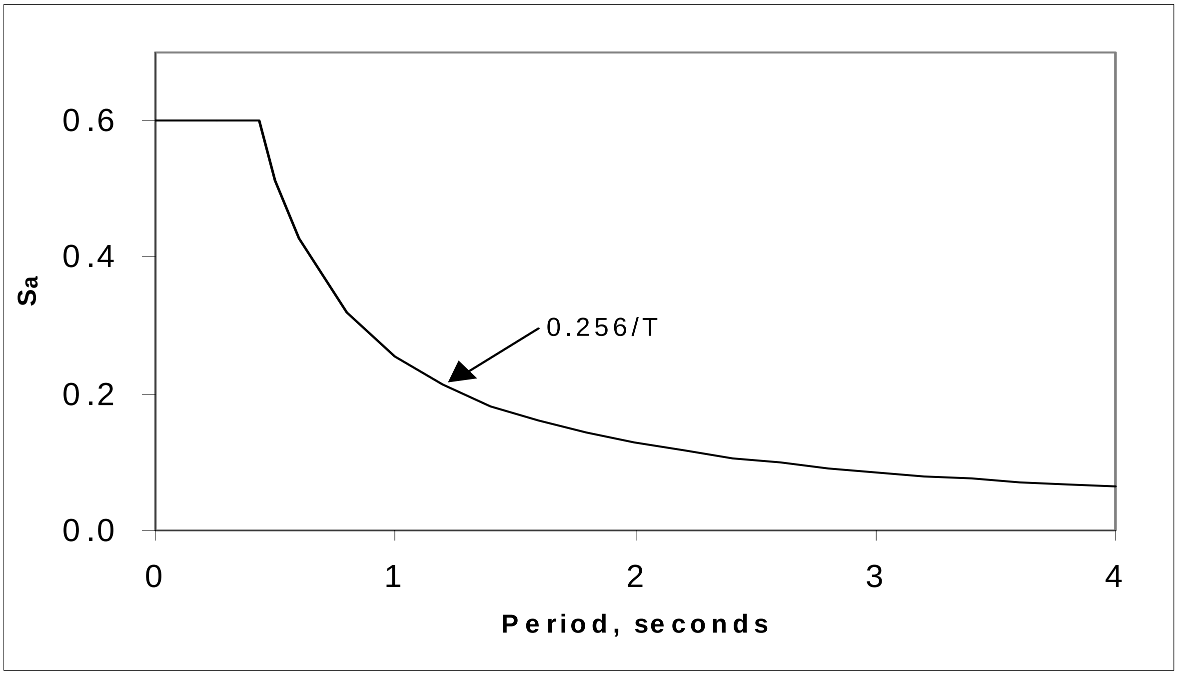
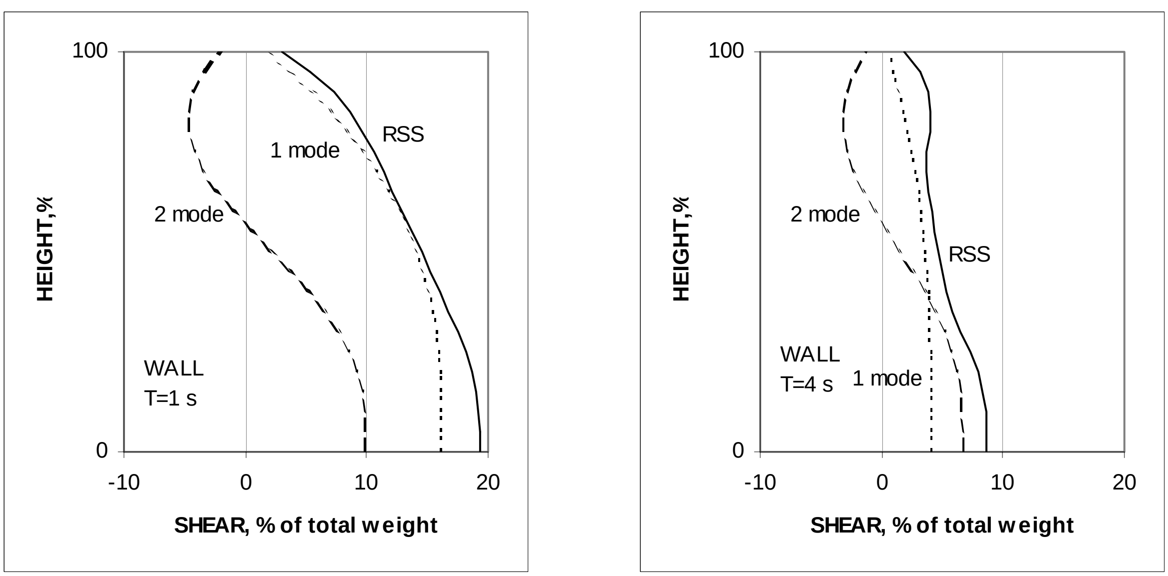
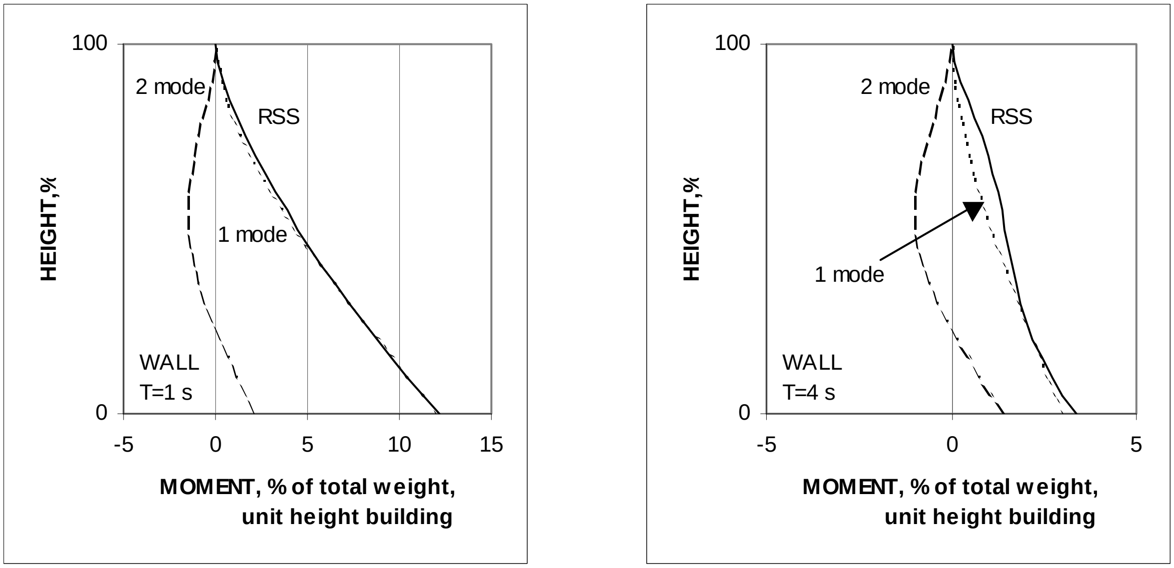


Fig. 4.4 Typical form of Design Response Spectrum



a) Wall system with T_1=1 sec b) Wall system with T_1=4 sec

Fig. 4.5 Example of Dynamic Shear Distributions (after D.L. Anderson)



b) Wall system with T_1=1 sec

b) Wall system with T_1=4 sec

Fig. 4.6 Example of Dynamic Moment Distributions (after D.L. Anderson)

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THE RESPONSE SPECTRUM

Appendices

Application of Response Spectrum in Structural Engineering

Additional Information

LIST OF APPENDICES

Appendix A: Dynamic properties of a 5 storey frame building (5 DOF) with uniform mass and stiffness distribution

Appendix B: Dynamic properties of a 5 storey frame building (5 DOF) with base isolation

Appendix C: Dynamic properties of a 4 storey frame building with light appendage at the roof level (5 DOF)

Appendix D: A note on dynamic analysis using the response spectrum method (from Computers & Structures)

Appendix E: A note on seismic analysis modeling to satisfy building codes (from Computers & Structures)

Appendix F: Selected examples of application of the response spectrum method (kindly provided by Prof. Mete Sozen and Prof. Luis Garcia of Purdue University)